

CONTRACTING ELEMENTS AND RANDOM WALKS

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ABSTRACT. We define a new notion of contracting element of a group and we show that contracting elements coincide with hyperbolic elements in relatively hyperbolic groups, pseudo-Anosovs in mapping class groups, rank one isometries in groups acting properly on proper $CAT(0)$ spaces, elements acting hyperbolically on the Bass-Serre tree in graph manifold groups. We also define a related notion of weakly contracting element, and show that those coincide with hyperbolic elements in groups acting acylindrically on hyperbolic spaces and with iwips in $Out(F_n)$, $n \geq 3$. We prove that any simple random walk in a non-elementary finitely generated subgroup containing a (weakly) contracting element ends up in a non-(weakly-)contracting element with exponentially decaying probability. Also, we show that each (weakly) contracting element is contained in a hyperbolically embedded elementary subgroup.

1. INTRODUCTION

A Morse element in a group G is an element h such that $H = \langle h \rangle$ is undistorted in G and any quasi-geodesic with endpoints on H stays within bounded distance from H , and the bound depends on the quasi-isometry constants only. Examples of Morse elements include infinite order elements in hyperbolic groups [Gr], hyperbolic elements (of infinite order) in relatively hyperbolic groups [DS], pseudo-Anosovs in mapping class groups [Be], etc. See [DMS] for further details and examples. Indeed, in all mentioned cases a stronger property than being Morse holds. Namely, all elements are *contracting* with respect to a suitable collection, called *path system*, of quasi-geodesic paths in the respective groups. Roughly speaking, the main property that a contracting element g satisfies is the existence of a map π_g from the group onto $\langle g \rangle$ such that if two points x, y have far away projections then all special paths from x to y pass close to $\pi_g(x)$ and $\pi_g(y)$. (This definition is less general than the one we will give, which is stated in terms of group actions on a metric space.) Related properties have been considered in [BFu1, Be, A-K, Si1]. These are the examples we can provide, see Section 3.

Theorem 1.1. *Let G be a relatively hyperbolic group (resp. mapping class group, group acting properly by isometries on a proper $CAT(0)$ space, graph manifold group). Then there exists a path system for G such that $g \in G$ is contracting if and only if g is hyperbolic (resp. is pseudo-Anosov, acts as a rank one isometry, acts hyperbolically on the Bass-Serre tree).*

We also define more general notions, that of being *weakly contracting*, which will be sufficient for our applications.

Theorem 1.2 (Proposition 3.7, Proposition 3.8). *Let G be a group acting acylindrically on a hyperbolic space X , and let \mathcal{PS} be the collection of all geodesics of X .*

Then $g \in G$ is weakly contracting for the weak path system (X, \mathcal{PS}) if and only if it acts hyperbolically on X .

There exists a weak path system for $\text{Out}(F_n)$, $n \geq 3$, such that $g \in \text{Out}(F_n)$ is weakly contracting if and only if it is iwip.

The disadvantage of the weak version of contractivity is that we were not able to show that weakly contracting elements are Morse (this is related to [DGO, Problem 9.4]).

We will treat all examples mentioned in Theorems 1.1 and 1.2 from a common perspective. A recent paper by Dahmani, Guirardel and Osin [DGO] has essentially the same aim. So, first of all, we will clarify the relation between the notion of contracting element and that of hyperbolically embedded subgroup defined in [DGO]:

Theorem 1.3 (Theorem 5.6). *Each weakly contracting element g is contained in a hyperbolically embedded elementary subgroup $E(g)$.*

The subgroup $E(g)$ is defined in a natural way in terms of projections, see Corollary 5.4. Notice that in view of Theorem 1.1 we obtain new examples of hyperbolically embedded subgroups, i.e. $E(g)$ where g acts as a rank one isometry on a proper $CAT(0)$ space, solving [DGO, Problem 9.5].

Our main result is that (weakly) contracting elements are generic, in the sense specified below. Let G be a group and $S \subseteq G$ a finite subset. Let $W_n(S)$ be the set of words of length n in the elements of S and their inverses. A *simple random walk* supported on $\langle S \rangle$ is a sequence of random variables $\{X_n\}$ taking values in G and with laws μ_n so that for each $g \in G$ and $n \in \mathbb{N}$ we have

$$\mu_n(\{g\}) = \frac{|\{w \in W_n(S) : w \text{ represents } g\}|}{|W_n(S)|}.$$

Theorem 1.4. *Let G be a group equipped with a path system (resp. weak path system) and let $H < G$ be a non-elementary finitely generated subgroup containing a contracting (resp. weakly contracting) element. Then the probability that a simple random walk supported on H gives rise to a non-contracting (resp. non-weakly-contracting) element decays exponentially in the length of the random walk.*

Roughly speaking, the theorem states that the probability that a long word written down choosing randomly generators of H represents a non-contracting element is very small, and in fact exponentially decaying in the length of the word.

For mapping class groups this was known already [Ma2], and related results can be found in [Ri1, Ri2, Ma1, MS]. We emphasize that we can give a self-contained proof of the theorem above modulo Lemma 6.3.

Outline. Section 2 contains the definitions we will use throughout. Section 3 is devoted to the proof of Theorems 1.1 and 1.2. Section 4 contains our main technical tool (Lemma 4.1), which gives us a way of constructing many contracting elements once we are given just one of them. In Section 5 we will show Theorem 1.3, and finally in Section 6 we will show our main result. In the Appendix we will sketch the proof of a “Weak Tits Alternative” not relying on Theorem 1.3 and [DGO].

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2. DEFINITIONS AND FIRST PROPERTIES

Let X be a metric space.

Definition 2.1. A path system \mathcal{PS} in X is a collection of (μ, μ) -quasi-geodesics in X , for some μ , such that

- (1) any subpath of a path in \mathcal{PS} is in \mathcal{PS} ,
- (2) all pairs of points in X can be connected by a path in \mathcal{PS} .

Elements of \mathcal{PS} will be called \mathcal{PS} -special paths, or simply special paths if there is no ambiguity on \mathcal{PS} .

Fix a path system \mathcal{PS} on the metric space X .

Definition 2.2. A subset $A \subseteq X$ (usually a \mathcal{PS} -special path) will be called \mathcal{PS} -contracting with constant C if there exists $\pi_A = \pi : X \rightarrow A$ such that

- (1) $d(\pi(x), x) \leq C$ for each $x \in X$,
- (2) for each $x, y \in X$, if $d(\pi(x), \pi(y)) \geq C$ then for any \mathcal{PS} -special path δ from x to y we have $d(\delta, \pi(x)), d(\delta, \pi(y)) \leq C$.

The map π will be called \mathcal{PS} -projection on A with constant C .

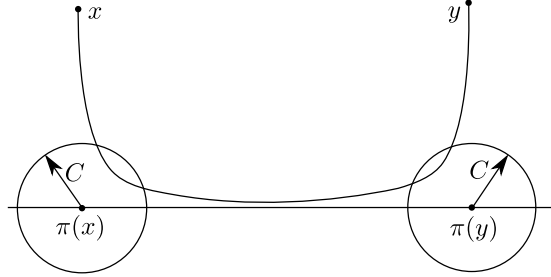


Figure 1

We point out a relation with properties that appeared in the literature in several contexts, see [BFu1, A-K, Si1].

Lemma 2.3. [Si1, Lemma 4.24] *Let X be a geodesic metric space and $A \subseteq X$. Denote by \mathcal{PS} the collection of all geodesics in X . Suppose that the map $\pi : X \rightarrow A$ satisfies the following properties for some C :*

- $d(x, \pi(x)) \leq d(x, A) + C$ for each $x \in X$,
- $\text{diam}(\pi(B_d(x))) \leq C$ for each $x \in X$, where $d = d(x, A)$.

Then π is a \mathcal{PS} -projection with constant depending on C only.

The following lemma will be used several times.

Lemma 2.4. *Let π be a \mathcal{PS} -projection with constant C on $A \subseteq X$. Then*

- (1) *whenever δ is a special path we have $\text{diam}(\pi(\delta)) \leq \text{diam}(\delta \cap N_C(A)) + 2C$ and more specifically $\text{diam}(\pi(\delta)) \leq C$ if $\delta \cap N_C(A) = \emptyset$.*

Also, there exists $k = k(\mathcal{PS})$ such that

- (2) *for each $x, y \in X$, $d(\pi(x), \pi(y)) \leq kd(x, y) + k$,*
- (3) *for each $x \in X$ we have $\text{diam}(\pi(B_r(x))) \leq C$ for $r = d(x, A)/k - k$,*
- (4) *for each $x \in X$, $d(x, \pi(x)) \leq kd(x, A) + k$.*

Proof. Item 1) is clear from the second projection property.

Item 2) follows from special paths being quasi-geodesics and the fact that either $d(\pi(x), \pi(y)) \leq C$ or any special path from x to y passes C -close to $\pi(x), \pi(y)$.

Item 3) holds in view of item 1) because, for c large enough, for each $y \in B_r(x)$ there is a special path connecting x to y and not intersecting $N_C(A)$.

In order to show item 4), consider a special path γ from x to some $y \in A$ with $d(x, y) \leq d(x, A) + 1$. If $d(y, \pi(x)) \leq C$, we are done. Otherwise, γ contains a point x' such that $d(x', \pi(x)) \leq C$. In particular,

$$d(x, \pi(x)) \leq d(x, x') + C \leq \mu d(x, y) + \mu + C \leq \mu d(x, A) + \mu + C + 1,$$

and we are done for c large enough. \square

Lemma 2.5. *If k is as in the previous lemma, $A, B \subseteq X$ are \mathcal{PS} -contracting subsets with constant C and $x \in X$, then*

$$\min\{d(\pi_A(x), \pi_A(B)), d(\pi_B(x), \pi_B(A))\} \leq kC + k + C.$$

Proof. If $\max\{d(\pi_A(x), \pi_A(B)), d(\pi_B(x), \pi_B(A))\} \leq kC + k + C$ there is nothing to prove, so, up to swapping A and B , suppose that $d(\pi_A(x), \pi_A(B)) > kC + k + C$. Let δ be a special path from x to some $x' \in B$, and let p be the first point in $\delta \cap N_C(A)$. If we show that there is no point $q \in \delta \cap N_C(B)$ before p , we are done by Lemma 2.4–(4) – (1). In fact, if there was such a point q , again by Lemma 2.4–(1) – (4) we would have

$$d(\pi_A(x), \pi_A(B)) \leq d(\pi_A(x), \pi_A(q)) + d(\pi_A(q), \pi_A(B)) \leq C + kC + k.$$

\square

We are ready for our main definitions.

Definition 2.6. Let G be a group. A *path system* (X, \mathcal{PS}) on the group G is a *proper* action of G on the metric space X preserving the path system \mathcal{PS} on X .

An infinite order element g of G will be called \mathcal{PS} -contracting if for some (and hence any) $x_0 \in X$ and any integer n

- (1) the orbit of x_0 is a quasi-geodesic,
- (2) all special paths connecting x_0 to $g^n(x_0)$ are \mathcal{PS} -contracting with uniform constant.

Recall from [BFu1] that, given a group G acting on the metric space X , the element $g \in G$ is *WPD* if the orbits of $\langle g \rangle$ are Morse quasi-geodesics and for every $R \geq 0$ and $x \in X$ there exists $N = N(R, x)$ such that

$$|\{h \in G : d(x, h(x)) \leq R, d(hg^N x, g^N x) \leq R\}| < +\infty.$$

The action of G on X will be said to be *WPD* if every $g \in G$ with Morse quasi-geodesic orbits is *WPD*. (This is not standard terminology as *WPD* actions are usually implied to be actions on hyperbolic spaces.)

We will need the following consequence of the *WPD* property.

Lemma 2.7. (cf. [BFu1, Proposition 6–(2)]) *Suppose that G acts on X and $g \in G$ is *WPD*. Then for each $x \in X$, $R \geq 0$ there exists L so that*

$$\{h \in G : d(x, hx) \leq R, \text{diam}(N_R(h\langle g \rangle x) \cap \langle g \rangle x) \geq L\}$$

is a finite set.

The meaning of the lemma is illustrated in Figure 2. Roughly speaking, the definition of WPD gives us that there are finitely many elements h so that for some large N we have the situation depicted for $N' = N$. The lemma gives us the same condition for all large enough N and without the requirement $N' = N$.

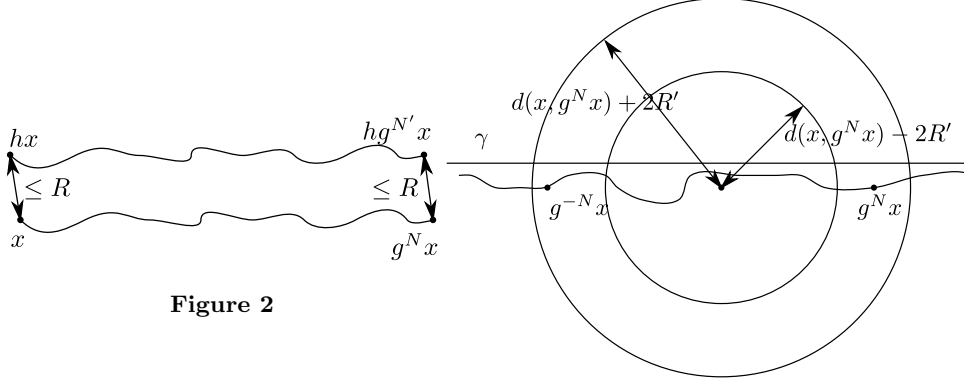


Figure 2

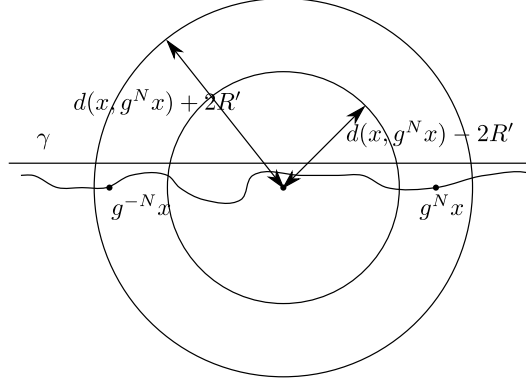


Figure 3

Proof. Fix x, R, g . First of all notice that, as $\langle g \rangle x$ is Morse, there exists $R' \geq R$ depending on R (and the data we fixed) so that if $d(x, hx) \leq R, d(g^N x, h\langle g \rangle x) \leq R$ then for each $0 \leq n \leq N$ we have $d(g^n x, h\langle g \rangle x) \leq R'$. We claim that it suffices to show that if

$$d(x, hx) \leq R, d(g^N x, hg^{N'} x) \leq R' \quad (*)$$

for some N, N' then either $d(g^N x, hg^N x) \leq C(R)$ or $d(g^N x, hg^{-N} x) \leq C(R)$, for some constant $C(R) \geq R$. In fact, if h, h' are as in the second case then hh' is easily seen to satisfy $d(hh'x, x) \leq 2R, d(g^N x, hh'g^N x) \leq 2C(R)$. So, the cardinality of the set as in the statement is at most the cardinality of

$$\{h \in G : d(x, hx) \leq 2R, d(hg^N x, g^N x) \leq 2C(R)\},$$

where for L large enough we can choose $N = N(C(R))$. By definition of $N(C(R))$ this set is finite, as required.

Now, suppose that $(*)$ holds. Then we have

$$|d(g^{N'} x, x) - d(g^N x, x)| \leq 2R'.$$

So, we just need to show that if N' satisfies this condition then $|N - N'| \leq K(R)$. In order to show this notice that, as the orbits of g are Morse quasi-geodesics, there exists a geodesic γ so that $d(g^N, \gamma), d(g^{N'}, \gamma), d(x, \gamma) \leq D$ (where D does not depend on N, N'), see Figure 3. In particular, if N' satisfies the condition then it is contained in a ball of radius, say, $2R' + 10D$ around $g^N x$ or in a ball of the same radius around $g^{-N} x$. The existence of $K(R)$ then follows from the orbit of g being a quasi-geodesic. \square

Definition 2.8. Let G be a group. A *weak path system* (X, \mathcal{PS}) on the group G is a WPD action of G on the metric space X which preserves the path system \mathcal{PS} on X .

An infinite order element g of G will be called weakly \mathcal{PS} -contracting if for some (and hence any) $x_0 \in X$ and any integer n

- (1) the orbit of x_0 is a quasi-geodesic,
- (2) all special paths connecting x_0 to $g^n(x_0)$ are \mathcal{PS} -contracting with uniform constant.

Remark 2.9. A proper action is WPD, and hence contracting elements are weakly contracting.

Lemma 2.10.

- (1) *Special paths which are \mathcal{PS} -contracting with constant C are Morse with constants depending on C only.*
- (2) *Contracting elements are Morse.*

Proof. The first part can be proven in the same (actually, slightly simpler) way as the second part, so we will spell out the proof for the second part only.

Let G, X, \mathcal{PS}, x_0 be as in the definition above, and let g be a contracting element. The fact that $\langle g \rangle x_0$ is a quasi-geodesic easily implies that $\langle g \rangle$ is a quasi-geodesic in G .

In order to show that g is Morse we now need to show the following. Let α be a (μ, c) -quasi-geodesic in G connecting e to g^n and let γ be a special path from x_0 to $g^n x_0$. Then for each $h \in \alpha$ we have $d(x_0, hx_0) \leq K$, where K depends on g, μ, c and the action of G . A constant depending on the said data will be referred to as universal.

Let π be a projection on γ with constant C . Increase C suitably and define $\pi' : G \rightarrow \langle g \rangle$ in such a way that $d(\pi'(x)x_0, \pi(x))$ is bounded by C . Let ρ be a universal constant such that

$$d(\pi'(h), \pi'(h')) \leq \rho d(\pi(hx_0), \pi(h'x_0)) + \rho$$

for each $h, h' \in G$. Let $r \in \mathbb{N}$ be the least integer so that $d(\alpha(j), \alpha(j+r)) > \mu^2(\rho C + 1) + c$ for each integer j such that $j, j+r$ are in the domain of α . Whenever $h = \alpha(j), h' = \alpha(j+r)$ we will say that h, h' are consecutive. Notice that there is a universal bound L on $d(hx_0, h'x_0)$ whenever $h, h' \in \alpha$ are consecutive. Suppose that $h_0, \dots, h_n \in \alpha$ is a maximal chain of consecutive points such that $d(hx_0, \gamma) \geq kL + k^2$, for k as in Lemma 2.4. Notice that we can bound the distance of h_0, h_n from γ again by a universal constant. We wish to show that n cannot be arbitrarily large. In fact, by Lemma 2.4–(3) we have $d(\pi(h_i(x_0)), \pi(h_{i+1}(x_0))) \leq C$, and hence $d(\pi(h_0(x_0)), \pi(h_n(x_0))) \leq nC$ so that $d(\pi'(h_0), \pi'(h_n)) \leq n\rho C + \rho$. For n large enough and in view of $d(h_i, h_{i+1}) > \mu^2(\rho C + 1) + c$ we get

$$d(h_0, h_n) \geq nr/\mu - c \geq \sum (d(h_i, h_{i+1}) - c)/\mu^2 - c >$$

$$n\rho C + \rho + d(h_0, \pi'(h_0)) + d(h_n, \pi'(h_n)) \geq d(h_0, h_n),$$

which is a contradiction. We can bound $d(p, \gamma)$ in terms of n, L, r , so we are done. \square

Remark 2.11. In view of the first part of the lemma in order to show that an element g is contracting it is enough to find C and a contracting special path γ_n with constant C connecting x_0 to $g^n x_0$ for each n . In fact, once this is done, any special path connecting x_0 to $g^n x_0$ will stay within controlled Hausdorff distance from γ_n .

3. EXAMPLES

3.1. Relatively hyperbolic group. Relatively hyperbolic groups were first introduced in [Gr] as a generalization of hyperbolic groups and have several different characterizations [Bo1, Fa, Da, Ya, Os1, DS, Dr]. There is also a characterization [Si1] in terms of projections on the peripheral subgroups, which is the one we will use.

Let G be a group hyperbolic relative to subgroups H_1, \dots, H_n . Let $\Gamma(G)$ be a Cayley graph of G with respect to a finite system of generators containing generators for each H_i . We will denote by $\Gamma(H_i)$ the subgraph of $\Gamma(G)$ corresponding to H_i . Consider disjoint copies of $\Gamma(H_i) \times \mathbb{R}$, one for each left coset of H_i in G , and glue to each $g\Gamma(H_i)$ the corresponding copy of $\Gamma(H_i) \times \mathbb{R}$ by identifying $\Gamma(H_i) \times \{0\}$ with $\Gamma(H_i)$. Denote the resulting space by $X = X(G)$ and notice that there is a natural action of G on X by isometries. Let \mathcal{PS} be the collection of all geodesics of X . We will say that an *infinite order* element of G is hyperbolic if it not conjugate into any H_i .

Proposition 3.1. *In the notation above, an element of G is contracting for the path system (X, \mathcal{PS}) if and only if it is hyperbolic.*

Proof. First of all, X is asymptotically tree-graded (i.e. metrically relatively hyperbolic) with respect to the glued copies of $\Gamma(H_i) \times \mathbb{R}$. This can be easily proven using the definition of asymptotic tree-gradedness in terms of asymptotic cones [DS]. Alternatively, one can apply either [MR] or [Si1, Theorem 0.2] (see the proof of [Si1, Corollary 5.31]). Recall that any hyperbolic element g is contained in an elementary subgroup $E(g)$ so that G is hyperbolic relative to $H_1, \dots, H_n, E(g)$, see [Os2]. In particular, for the same reasons as above, given any hyperbolic element g we can add $E(g)x_0$ to the collection of peripheral subsets of X , for some $x_0 \in X$. By the definition of relative hyperbolicity given in [Si1] there exists a projection on $E(g)x_0$, and hence on all geodesics connecting $g^{-n}x_0$ to g^nx_0 .

Finally, observe that thanks to the “thickening” of the peripheral subgroups, any non-hyperbolic element is clearly not contracting. \square

3.2. Mapping class group. In this subsection we will assume familiarity with the notion of curve complex, marking complex and hierarchy paths. We will use results from [MM1, MM2], see also [Bo2].

Proposition 3.2. *Let $\mathcal{M}(S)$ be the marking complex of the closed orientable punctured surface S and let $\mathcal{H}(S)$ be the collection of all subpaths of hierarchy paths. An element of $MCG(S)$ is contracting for the path system $(\mathcal{M}(S), \mathcal{H}(S))$ if and only if it is pseudo-Anosov.*

Remark 3.3. We remark that in view of the proof of [BFu1, Proposition 8.1] (see the sixth to last line) and Lemma 2.3 the collection of all geodesics in Teichmüller space endowed with the Weil-Peterson metric has the property that all axes of pseudo-Anosov elements are contracting.

It is well known that hierarchy paths are quasi-geodesics with uniform constant, and that any non-pseudo-Anosov element is not Morse as it has infinite index in its centralizer.

Recall from [Be] that a pair of points $(\mu_1, \mu_2) \in \mathcal{M}(S)$ is D -transverse if for each $Z \subsetneq S$ we have $d_{C(Z)}(\mu_1, \mu_2) \leq D$. The proposition follows directly from the

lemma below and the fact that pairs of points lying on the orbit of a pseudo-Anosov element are D -transverse.

Lemma 3.4. *For each D -transverse pair μ_1, μ_2 each hierarchy path $[\mu_1, \mu_2]$ is $\mathcal{H}(S)$ -contracting with constant $C = C(D)$.*

Proof. Let B be such that if x, y lie on the (subpath of a) hierarchy path $[p, q]$ with main geodesic H then

- (1) for each subsurface $Y \subseteq S$ we have $d_{C(Y)}(x, y) \leq d_Y(p, q) + B$,
- (2) $d_{C(S)}(x, H) \leq B$,
- (3) for any $\gamma \in H$ there exists z such that $d_{C(S)}(x, z) \leq B$.

Define $\pi : \mathcal{M}(S) \rightarrow [\mu_1, \mu_2]$ as follows. Consider $\mu \in \mathcal{M}(S)$ and let $\gamma_\mu = \pi_S(\mu)$. Pick $\alpha_\mu \in H$ (the main geodesic for the given hierarchy path) such that $d_{C(S)}(\gamma_\mu, \alpha_\mu) = d_{C(S)}(\gamma_\mu, H)$. Finally, choose $\nu \in [\mu_1, \mu_2]$ in such a way that $\pi_S(\nu)$ is B -close to α_μ and set $\pi(\mu) = \nu$.

Pick any $\nu_1, \nu_2 \in \mathcal{M}(S)$. Suppose $d(\pi(\nu_1), \pi(\nu_2)) \geq C(D)$, where $C(D)$ will be determined later. Notice that $d_{C(S)}(\alpha_{\nu_1}, \alpha_{\nu_2})$ is bounded from below by a linear function of $C(D)$, by the distance formula and D -transversality, so that for $C(D)$ large enough we can assume $d_{C(S)}(\alpha_{\nu_1}, \alpha_{\nu_2}) \geq 100\delta$, where δ is the hyperbolicity constant of $C(S)$. Consider a hierarchy path $[\nu_1, \nu_2]$ (more precisely containing ν_1, ν_2 , but this does not affect what follows) and let H' be the main geodesic. We have that H' contains points β_1, β_2 which are B -close to the projections of $\nu'_1, \nu'_2 \in [\nu_1, \nu_2]$ on $C(S)$ and such that $d_{C(S)}(\alpha_{\nu_i}, \beta_i) \leq 10\delta$.

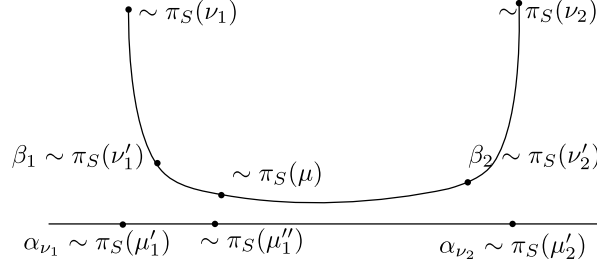


Figure 4

Our aim is to show that any point μ on $[\nu'_1, \nu'_2]$ such that $20\delta + 2B + 10 \leq d_{C(S)}(\nu, \beta_1) \leq 20\delta + 3B + 20$ has the property that $d(\mu, \pi(\nu_1)) \leq C(D)$ (an analogous property holds for ν_2).

In order to show this, we will now analyze the K -large domains for the pair ν'_1, ν'_2 , where $K > 3(D + B)$ (a L -large domain for a pair of markings ρ_1, ρ_2 is a subsurface $Y \subseteq S$ so that $d_{C(Y)}(\rho_1, \rho_2) \geq L$). First, we claim that there are no common $(K/3)$ -large domains for the pairs μ'_1, ν'_1 and μ'_2, ν'_2 , where $\mu'_i \in [\mu_1, \mu_2]$ projects B -close to α_{ν_i} .

In fact, any $(K/3)$ -large domain X_i for the pair μ'_i, ν'_i is contained in some $S \setminus \gamma_i$, where γ_i is a simple closed curve appearing in a hierarchy path connecting $\pi_S(\mu'_i)$ and ν'_i . This implies that $X_1 \neq X_2$ as $d_{C(S)}(\gamma_1, \gamma_2) \geq 3$ for $C(D)$ large enough.

We can now use the fact that there are no D -large domains for the pair μ'_1, μ'_2 . Suppose $d_{C(Y)}(\nu'_1, \nu'_2) \geq K$, for some subsurface $Y \subsetneq S$. Then (at least) one of the following must hold: $d_{C(Y)}(\nu'_1, \mu'_1) \geq K/3$, $d_{C(Y)}(\mu'_1, \mu'_2) \geq K/3$ or $d_{C(Y)}(\nu'_1, \mu'_1) \geq$

$K/3$. However, the second inequality does not hold by hypothesis. Hence, any K -large domain for ν'_1, ν'_2 , and so any $(K+B)$ -large domain for any pair of points on $[\nu'_1, \nu'_2]$, is a $(K/3)$ -large domain for μ'_1, ν'_1 or μ'_2, ν'_2 (but not both).

With a similar argument we get that for each $\nu \in [\nu'_1, \nu'_2]$ and each $\mu \in [\mu'_1, \mu'_2]$ all $(4K/3 + D + 2B)$ -large domains $Y \subsetneq S$ for μ, ν are $(K/3)$ -large domains for μ'_1, ν'_1 or μ'_2, ν'_2 . Choosing ν as above we have that $\pi_S(\nu)$ is far enough from β_1 and β_2 to guarantee that no $(4K/3 + D + 2B)$ -large domain $Y \subsetneq S$ for ν, μ is a K -large domain for μ'_i, ν'_i , where $\mu \in [\mu'_1, \mu'_2]$ is such that $20\delta + 2B + 10 \leq d_{C(S)}(\mu, \mu'_i) \leq 20\delta + 3B + 20$. By the distance formula $d(\nu, \mu'_1)$ can be bounded in terms of δ and B , so we are done. \square

3.3. Graph manifolds. A graph manifold is a compact connected 3-manifold (possibly with boundary) which admits a decomposition into Seifert fibred surfaces, when cut along a collection of embedded tori and/or Klein bottles. In particular a graph manifold is a 3-manifold whose geometric decomposition admits no hyperbolic part.

Let M be a graph manifold. It is known [MKL] that its universal cover \widetilde{M} is bilipschitz equivalent to the universal cover \widetilde{N} of a *flip* graph manifold N , that is to say a graph manifold with some special properties (among which a metric of nonpositive curvature). We will not need the exact definition of such manifolds, but we need to know that we can choose a bilipschitz equivalence $\phi : \widetilde{M} \rightarrow \widetilde{N}$ that preserves the geometric components. In [Si2], a family of paths $\mathcal{PS}(N)$ in \widetilde{N} has been defined, and those paths satisfy the following:

Lemma 3.5.

- (1) All paths in $\mathcal{PS}(N)$ are bilipschitz, with controlled constant;
- (2) any subpath of a path in $\mathcal{PS}(N)$ is again in $\mathcal{PS}(N)$;
- (3) if for $i = 1, 2$ α_i is a special path connecting some point in X_{w_i} to some point in $X_{w'_i}$ and the vertex v
 - lies on the geodesic connecting w_i, w'_i , and
 - $d(v, w_i), d(v, w'_i) \geq 2$,
then $\alpha_1 \cap X_v = \alpha_2 \cap X_v$.

Let $\mathcal{PS}(M) = \{g\phi^{-1}(\gamma) : g \in \pi_1(M), \gamma \in \mathcal{PS}(N)\}$.

Proposition 3.6. $(\widetilde{M}, \mathcal{PS}(M))$ is a path system for $\pi_1(M)$. An element of $\pi_1(M)$ is contracting if and only if it acts hyperbolically on the Bass-Serre tree of M .

Proof. The first part follows directly from Lemma 3.5–(1) – (2), so we can focus on the second part.

Let us prove the “if” part. Let $g \in \pi_1(M)$ be an element acting hyperbolically on the Bass-Serre tree. Let $x_0 \in X_v$ for some vertex v of the Bass-Serre tree such that $\{g^i v\}$ is contained in a bi-infinite geodesic. For each n let γ_n be a special path from x_0 to $g^n(x_0)$ and define $\pi_n : \widetilde{M} \rightarrow \gamma_n$ in the following way. Let π'_n be the projection in the Bass-Serre tree on $[v, g^n v]$, and define $v(x)$ for each $x \in \widetilde{M}$ to be a vertex in the Bass-Serre tree such that $x \in X_{v(x)}$. Also, let $\pi''_n(x) \in \{v, \dots, g^n v\}$ so that $d(\pi'(x), \pi''(x)) \leq d(v, gv)/2$. Finally, define $\pi_n(x)$ to be a point in $X_{\pi''_n(x)} \cap \gamma_n$.

Now, suppose that we have $x, y \in \widetilde{M}$ such that $d(\pi_n(x), \pi_n(y))$ is large enough. Then we can conclude that, say, $d(\pi'(v(x)), \pi'(v(y))) \geq 10$. We will now use Lemma 3.5 and the fact that ϕ preserves the geometric components to find a bound on

the distance between any special path δ connecting x, y and $\pi_n(x), \pi_n(y)$. By hypothesis, ϕ induces a simplicial isomorphism from the Bass-Serre tree of M to that of N , which we will still denote by ϕ . Suppose that $\delta = h\phi^{-1}(\alpha)$, $\gamma_n = h_n\phi^{-1}(\alpha_n)$. Then it is clear from Lemma 3.5 that δ shares a subpath with any special path $h\phi^{-1}(\beta)$ such that β connects some point in $X_{\phi(h^{-1}(v(x)))}$ to some point in $X_{\phi(h^{-1}(v(y)))}$. More precisely, those paths coincide in X_w whenever w is “well within” $[\pi'(v(x)), \pi'(v(y))]$. We can choose β so that the endpoints of $h^{-1}\phi(\beta)$ are $x_0, g^n x_0$, and then the fact that γ_n is Morse with constant not depending on n [MKL] clearly implies the desired bound.

The “only if” part is easy. In fact, if $g \in \pi_1(M)$ is conjugate to some element in a vertex group, then it is clearly not even Morse in view of the fact that such groups are virtually products of a free group and \mathbb{Z} . \square

3.4. Groups acting acylindrically on hyperbolic spaces and $Out(F_n)$. Recall that a group G is said to act *acylindrically* on the metric space X if for all d there exist R_d, N_d so that whenever $x, y \in X$ satisfy $d(x, y) \geq R_d$ we have

$$|\{g \in G : d(x, gx) \leq d, d(y, gy) \leq d\}| \leq N_d.$$

Graph manifold groups act acylindrically on their Bass-Serre tree. However, we are not able to prove the equivalent of Proposition 3.6 for groups acting acylindrically on trees or hyperbolic spaces, and indeed it might not be true. Nonetheless, we have the following.

Proposition 3.7. *Let G be a group acting acylindrically (by isometries) on the hyperbolic space X and let \mathcal{PS} be the collection of all geodesics of X . Then $g \in G$ is weakly contracting for the weak path system (X, \mathcal{PS}) if and only if it acts hyperbolically on X .*

Proof. Given the following easy fact, the proof just requires unwinding the definitions: if X is hyperbolic then all geodesics in X are contracting with the same constant. This is, in other forms, well-known (and it follows from more general results in [Si1]). Here is a proof for the convenience of the reader. Let $\gamma \subseteq X$ be a geodesic, and let $\pi : X \rightarrow \gamma$ be any map so that $d(x, \pi(x)) \leq d(x, \gamma) + 1$ for each $x \in X$. Suppose that, for some x, y , we have $d(\pi(x), \pi(y)) \geq 8\delta + 3$, where δ is the hyperbolicity constant of X , and let α be a geodesic from x to y . As quadrangles in X are 2δ -thin (see, e.g. [Gr]), we have that $[\pi(x), \pi(y)]$ is contained in the 2δ -neighborhood of $\alpha \cup [x, \pi(x)] \cup [y, \pi(y)]$. Due to our choice of π , points $[\pi(x), \pi(y)]$ farther than $4\delta + 1$ from the endpoints are not 2δ -close to $[x, \pi(x)] \cup [y, \pi(y)]$. Hence, there is a point on α which is 2δ -close to the point on $[\pi(x), \pi(y)]$ at distance $4\delta + 2$ from $\pi(x)$, and so we have $d(\alpha, \pi(x)) \leq 6\delta + 2$. \square

Bestvina and Feighn proved that the complex of free factors for $Out(F_n)$ is hyperbolic, that the action of $Out(F_n)$ on it satisfies WPD and that $g \in Out(F_n)$ acts hyperbolically if and only if it is irreducible with irreducible powers (iwip) [BFe2, Theorem 8.3]. In particular we have the following.

Proposition 3.8. *Let X be the complex of free factors of $Out(F_n)$ for some $n \geq 3$. Then $g \in Out(F_n)$ is weakly contracting for the weak path system (X, \mathcal{PS}) if and only if it is iwip.*

Remark 3.9. For our applications we could also use the complexes constructed in [BFe1].

The present state of knowledge about the geometry of $Out(F_n)$ is not advanced enough to attempt imitating what we have done, say, for mapping class groups. It is not even known whether there is an algebraic characterization of Morse elements along the lines of the other cases, and hence the following is perhaps the most basic question arising at this point.

Question 3.10. Is it true that all Morse elements of $Out(F_n)$ for $n \geq 3$ are iwip?

The converse is true by [A-K]. A standard way to show that an (infinite order) element is not Morse is to show that (the cyclic group generated by) it has infinite index in its centralizer. There might be infinite order non-iwip elements of $Out(F_n)$ which have finite index in their centralizers. So, if the answer to the previous question is negative, one might ask the following.

Question 3.11. Is it true that all elements of $Out(F_n)$ that have finite index in their (virtual) centralizers are Morse?

We remark that even if the answer to the first question is negative it might still be possible to apply our methods to find out something about iwip's, using a trick similar to the one we used for relatively hyperbolic groups, for example.

3.5. Groups acting properly on proper $CAT(0)$ spaces. Recall that an isometry of a $CAT(0)$ space is said to be rank one if it is hyperbolic and some (equivalently, every) axis of g does not bound a half-flat.

Remark 3.12. In the case when G is a right-angled Artin group and X its standard $CAT(0)$ cube complex, the elements of G that act as rank 1 isometries coincide with those not conjugated into a join subgroup, see [BC].

Proposition 3.13. *Let G be a group acting properly by isometries on the proper $CAT(0)$ space X and let \mathcal{PS} be the collection of all geodesics in X . Then $g \in G$ is contracting for the path system (X, \mathcal{PS}) if and only if it acts as a rank one isometry.*

Proof. It is clear that (X, \mathcal{PS}) is a path system on G , and the “only if” part follows from the fact that if g does not act as a rank one isometry then it is not Morse. Suppose that g acts as a rank one isometry. By [BFu1, Theorem 5.4] the closest point projection on an axis l of g has the property that there exists B so that each ball disjoint from l projects onto a set of diameter at most B . The same is true for subsegments of the axis as well, see [BFu1, Lemma 3.2]. Property 2) in Definition 2.2 then follows from Lemma 2.3. \square

4. CONSTRUCTION OF CONTRACTING ELEMENTS

Let (X, \mathcal{PS}) be a weak path system on the group G . Let $x_0 \in X$ be any point and let γ be a special path connecting x_0 to gx_0 . Consider a projection π_γ on γ with constant C (see Definition 2.2). We do not assume that g is (weakly) contracting, but it will always be in applications. For $h \in G$ and $x \in X$, define $\pi_{h\gamma}(x) = h\pi_\gamma(h^{-1}(x))$ (this is clearly a projection on $h\gamma$ with constant C). Notice that for each $h_1, h_2 \in G$ and $x, y \in X$ we have $h_1\pi_{h_2\gamma}(x) = \pi_{h_1h_2\gamma}(h_1(x))$ so that

$$d(\pi_{h_2\gamma}(x), y) = d(\pi_{h_1h_2\gamma}(h_1(x)), h_1y).$$

This property will be referred to as equivariance.

Lemma 4.1. *Fix the notation above. There exist $K = K(\mathcal{PS}, C)$ with the following property. Suppose that $h \in G$ is such that*

- (1) $d(x_0, gx_0) - [d(\pi_\gamma(ghx_0), gx_0) + d(\pi_\gamma(h^{-1}x_0), x_0)] \geq K$.
 (2) $\text{diam}(\pi_\gamma(gh\gamma)) \leq C, \text{diam}(\pi_{gh\gamma}(\gamma)) \leq C$.

Then hg is weakly contracting. In particular, if the action of G on X is proper then hg is contracting.

Proof. Set $x_i = (hg)^i(x_0)$, $x'_i = (hg)^{i-1}h(x_0)$ for each integer i .

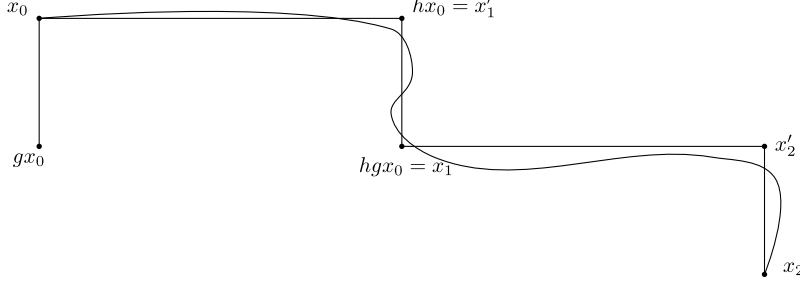


Figure 5

Step 1. Structure of the orbit. We claim that, when K is large enough, any special path from x_i to x_j passes uniformly close to each x_k and x'_k for $\min\{i, j\} + 1 \leq k \leq \max\{i, j\}$. This clearly implies that the orbit $\{x_i\}$ of $\langle hg \rangle$ is a quasi-geodesic. It is enough to show the claim for $i = 0$, and we will assume $j \geq 0$, the proof for $j \leq 0$ being analogous.

Set $d_1 = d(\pi_\gamma(h^{-1}x_0), x_0)$, $d_2 = d(\pi_\gamma(ghx_0), gx_0)$, so that (1) can be rewritten as $d(x_0, gx_0) \geq d_1 + d_2 + K$. For convenience, we increase C so that Lemma 2.5 holds for C instead of $kC + k + C$. We will show by induction on j that

- (*)_j $d(\pi_{(hg)^{k-1}h\gamma}(x_0), x'_k) \leq d_1 + 2C$, and
 (***)_j $d(\pi_{(hg)^{k-1}h\gamma}(x_j), x_k) \leq d_2 + 2C$ for each $1 \leq k \leq j$.

This implies our claim by the definition of \mathcal{PS} -contracting and the hypothesis for K large enough.

(*)_1 holds by the definition of d_1 (and equivariance), while (***)_1 holds as $x_1 \in h\gamma$ and (***)_2 by the definition of d_2 . Suppose that (*_j), (***)_j hold. In view of equivariance, in order to show (*_{j+1}) we only need to prove $d(\pi_{(hg)^j h\gamma}(x_0), x'_{j+1}) \leq d_1 + 2C$. We wish to apply Lemma 2.5 with $A = (hg)^{j-1}h\gamma$, $B = (hg)^j h\gamma$. By Step 1, $\text{diam}(\pi_A(B)) \leq C$. Combining this with $d(\pi_A(x_0), x'_j) \leq d_1 + 2C$ (a special case of (*_j)), $d_2 = d(\pi_A(x_{j+1}), x_j)$ and (1) we get

$$d(\pi_A(x_0), \pi_A(B)) \geq$$

$$d(x'_j, x_j) - d(\pi_A(x_0), x'_j) - \text{diam}(\pi_A(B)) - d(\pi_A(x_{j+1}), x_j) \geq K - 2C.$$

So, for K large enough and in view of Lemma 2.5, we get $d(\pi_B(x_0), \pi_B(A)) \leq C$. As $\text{diam}(\pi_B(A)) \leq C$ and $d(x'_{j+1}, \pi_B(A)) \leq d_1$ by the definition of d_1 and equivariance, we get the desired inequality.

Let us now show (**_{j+1}). Using equivariance and the inductive hypothesis we can deal with all cases with $k \geq 2$, so we only need to show $d(\pi_{h\gamma}(x_{j+1}), hg x_0) \leq d_2 + C$. The proof is very similar to that of (*_j), so we will just sketch it. We set $A = \gamma$, $B = hgh\gamma$, and notice that $d(\pi_B(A), \pi_B(x_{j+1}))$ is large, so that by Lemma 2.5 we have that $d(\pi_A(x_{j+1}), \pi_A(B))$ is small. As $d(hg x_0, \pi_A(B)) \leq d_2$ and $\pi_A(B)$ has bounded diameter, we are done.

Step 2. Contracting property. We have to provide projections with uniform constant on any special path connecting x_0 to x_n . Pick such a path δ , for some $n \geq 0$ (the proof for $n \leq 0$ is analogous). Let $x \in X$. Define $\pi(x) = \pi_\delta(x)$ as follows. Set

$$m(x) = \min\{k \leq n : d(\pi_{(hg)^{k-1}h\gamma}(x), \pi_{(hg)^{k-1}h\gamma}(x_0)) \leq C\},$$

and let $\pi(x)$ be a point on δ at distance at most $d_1 + C$ from $x'_{m(x)}$.

The first property is clear from the previous step, as one can easily reduce to check it just for the x_j 's.

Let us now show the second projection property. Suppose that $|m(x) - m(y)| \geq 10$. Set $A = (hg)^{m(x)-1}h\gamma$, $B = (hg)^{m(y)-1}h\gamma$. Up to swapping x and y we can assume $m(y) > m(x)$. In particular

$$d(\pi_{(hg)^{m(x)-1}h\gamma}(x), \pi_{(hg)^{m(x)-1}h\gamma}(y)) > C,$$

so that any special path α from x to y passes close to $\pi_{(hg)^{m(x)-1}h\gamma}(x)$, which is by the previous step and the definition of π close to $\pi(x)$. It can be shown similarly that α passes close to $\pi(y)$. \square

The following variation will be used in the Appendix.

Lemma 4.2. *There exist $K = K(\mathcal{PS}, C)$ with the following property. Suppose we are given a contracting set A with constant C and $g, h \in G$ with $x_0, gx_0 \in A$ such that*

- (1) $d(x_0, gx_0) - [d(\pi_A(gx_0), gx_0) + d(\pi_A(h^{-1}x_0), x_0)] \geq K$.
- (2) $\text{diam}(\pi_A(gxA)) \leq C$, $\text{diam}(\pi_{ghA}(A)) \leq C$.

Then hg is contracting.

Proof. Notice that we never used in the proof above that γ is a special path (we just used the properties of a projection on γ). Hence, as we now assume Step 1, the proof of Steps 2 and 3 can be followed verbatim to prove the lemma. \square

5. CONTRACTING IMPLIES HYPERBOLICALLY EMBEDDED

When γ is a special path and $x, y \in \gamma$, we will write $x < y$ if for all pre-images x', y' under γ of x, y we have $x' < y'$.

Lemma 5.1 (Projections are coarsely monotone). *Let γ, δ be special paths, assume that γ is a (μ, μ) -quasi-geodesic and let π_γ be a projection on γ with constant C . There exists $K = K(\mu, C)$ with the following property. Suppose, for $i = 0, 1, 2$, that $y_i = \pi_\gamma(x_i)$, for some $x_i \in \delta$. If $y_0 < y_1 < y_2$ and $d(y_0, y_1), d(y_1, y_2) \geq K$, then either $x_0 < x_1 < x_2$ or $x_2 < x_1 < x_0$.*

Proof. Suppose by contradiction that $x_0 < x_2 < x_1$ (the other cases can be dealt with in the same way). Then we can find points $x, y \in \delta$ lying, respectively, between x_0 and x_2 and between x_2 and x_1 such that $d(x, y_1), d(y, y_1)$ is bounded in terms of C (we use Lemma 2.10–(1) here). Also, $d(x_2, x), d(x_2, y)$ can be made arbitrarily large up to increasing K in view of Lemma 2.4–(2). This gives a contradiction with γ being a quasi-geodesic, see Figure 6. \square

Let (X, \mathcal{PS}) be a weak path system on the group G , and let $x_0 \in X$.

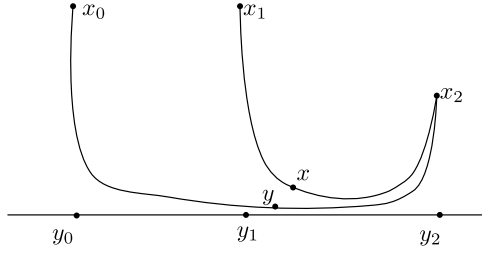


Figure 6

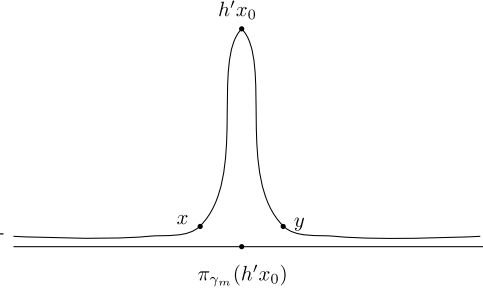


Figure 7

Lemma 5.2. *Suppose that $g \in G$ is weakly contracting and set $H_0 = \langle g \rangle$. Denote by γ_m a special path connecting $g^{-m}x_0$ to $g^m x_0$. Then there exists K depending only on g such that for each $h \in G$ and $m \in \mathbb{N}$ either $\pi_{\gamma_m}(hH_0x_0)$ has diameter bounded by K or it is K -dense in γ_m . Moreover, there exists K' so that if $d(x_0, H_0x_0) \geq K'$ then the first case holds.*

Proof. If H_0 is contained in a neighborhood of hH_0 of finite radius, then it is easily seen that the second case holds. We can then find $h' \in hH_0$ such that $h'x_0$ is as far as we wish from H_0x_0 . We claim that if $d(h'x_0, H_0x_0)$ is large enough, then $\pi_{\gamma_m}(h'\gamma_m)$ cannot contain points on both sides of $\pi_{\gamma_m}(h'x_0)$ and arbitrarily far from it. In fact, such points would be by the previous lemma projections of points on opposite sides of $h'x'_0$, where $x'_0 \in \gamma_m$ is close to x_0 , and hence one can find $x, y \in h'\gamma_m$ with $x < h'x_0 < y$ such that $d(x, \pi_{\gamma_m}(h'x_0)), d(y, \pi_{\gamma_m}(h'x_0)) \leq C$, where C is the projection constant. This is easily seen to contradict $h'\gamma_m$ being a quasi-geodesic, see Figure 7.

Now, if $\text{diam}(\pi_{\gamma_m}(hH_0))$ is large enough we can find (see Figure 8) a large family $\{N_i\}$ of integers such that

- $\pi_{\gamma_m}(g^{N_i}h'\gamma_m)$ has positive Hausdorff distance from $\pi_{\gamma_m}(g^{N_j}h'\gamma_m)$ whenever $i \neq j$. This gives in particular that $g^{N_i}h'$ and $g^{N_j}h'$ do not belong to the same left coset of H_0 whenever $i \neq j$.
- $g^{N_i}h'\gamma_m$ contains a point in some ball of fixed radius around $g^{m_0}x_0$, as well as a long subpath contained in a neighborhood of γ_m of radius depending on the projection constant and the quasi-geodesic constants of special paths only. This implies, by Lemma 2.7, that all left cosets $g^{N_i}h'H_0$ contain an element from a certain fixed finite subset of G .

Therefore there is a bound on the cardinality of $\{N_i\}$, and hence a bound on $\text{diam}(\pi_{\gamma_m}(hH_0))$. \square

One can similarly prove the following.

Lemma 5.3. *Suppose that $H < G$ is not virtually cyclic and it contains the weakly contracting element g . Then for each K there exists a left coset $hH_0 \subseteq H$ of $H_0 = \langle g \rangle$ such that $d(hH_0x_0, H_0x_0) \geq K$.*

Proof. As H is not virtually cyclic, it is well-known that it is not quasi-isometric to \mathbb{Z} . In particular, we can find h' such that h' is as far as we wish from H_0 . So, either $d(h'x_0, H_0x_0)$ can be made arbitrarily large, and in this case we have $\text{diam}(\pi_{\gamma_m}(h'\gamma_m)) \leq K$ by the “moreover” part of Lemma 5.2, or we can use Lemma

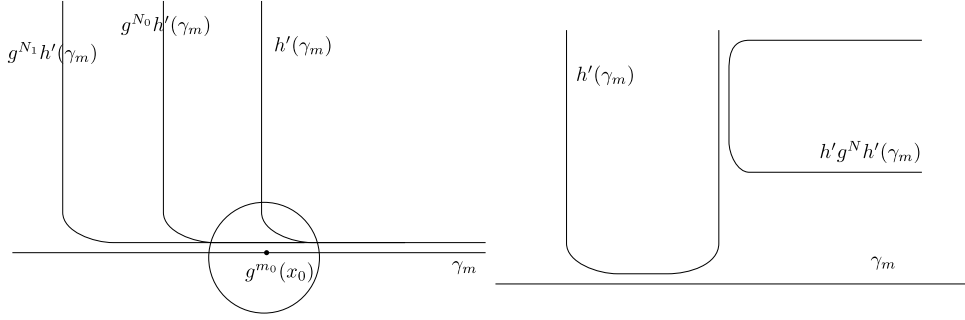


Figure 8

Figure 9

2.7 to get that $\pi_{\gamma_m}(h'\gamma_m)$ is not K -dense in γ_m for each m , and hence once again $\text{diam}(\pi_{\gamma_m}(h'\gamma_m)) \leq K$ by Lemma 5.2. It is then easily seen that we can choose h of the form $h'g^N h'$ with N large enough so that $\pi_{h'\gamma_m}(\gamma_m)$ is far from $\pi_{h'\gamma_m}(h\gamma_m)$, see Figure 9. \square

As a consequence of the previous lemmas we get:

Corollary 5.4. *Each weakly contracting element is contained in a virtually cyclic subgroup, denoted $E(g)$, such that there exists a uniform bound on $\pi_{\gamma_m}(hE(g))$ for each $h \notin E(g)$ and m .*

Proof. Just define $E(g)$ to be the collection of all h such that $\pi_{\gamma_m}(h\langle g \rangle)$ has Hausdorff distance from γ_m uniformly bounded in m . This is clearly a subgroup containing g , and it is virtually cyclic by the previous lemma. The uniform bound is a consequence of Lemma 5.2. \square

We point out some extra properties of $E(g)$.

Lemma 5.5. *There exists a positive integer n such that for each $h \in E(g)$ we either have $hg^n h^{-1} = g^n$ or $hg^n h^{-1} = g^{-n}$. In particular, for each integer k , $\langle g^{nk} \rangle$ is normal in $E(g)$ and $E(g)$ is the centralizer of g^{2n} .*

Proof. Choose representatives h_1, \dots, h_k of the left cosets of $\langle g \rangle$ in $E(g)$. Choose $n_i > 0$ so that $h_i g^{n_i} h_i^{-1} \in \langle g \rangle$. Such n_i exists because there are finitely many left cosets of $\langle g \rangle$, hence one can find $k_i > m_i$ such that $g^{k_i} h_i^{-1}$ and $g^{m_i} h_i^{-1}$ belong to the same left coset and set $n_i = k_i - m_i$. Choose $n = \prod n_i$. Then it is readily seen that $hg^n h^{-1} \in \langle g \rangle$ for each $h \in E(g)$. In particular $hg^n h^{-1} = g^k$ for some k , and we would like to prove $k = \pm n$. Notice that there exists j such that $h^j \in \langle g \rangle$. Hence

$$g^{n^j} = h^j g^{n^j} h^{-j} = g^{k^j}.$$

In particular $n^j = k^j$, hence $k = \pm n$, as required. \square

Theorem 5.6. *Let G be any group endowed with a fixed weak path system. For each weakly contracting element $g \in G$, $E(g)$ is hyperbolically embedded in G .*

We will use work of Bestvina, Bromberg and Fujiwara [BBF], similarly to the proof of [DGO, Theorem 4.39]. We will now give the relevant definitions and state the result we need. Let \mathbf{Y} be a set and for each $Y \in \mathbf{Y}$ let $\mathcal{C}(Y)$ be a geodesic

metric space. For each Y let $\pi_Y : \mathbf{Y} \setminus \{Y\} \rightarrow \mathcal{P}(\mathcal{C}(Y))$ be a function (where $\mathcal{P}(Y)$ is the collection of subsets of Y). Define

$$d_Y^\pi(X, Z) = \text{diam}\{\pi_Y(X) \cup \pi_Y(Z)\}.$$

Using the enumeration in [BBF], consider the following Axioms:

- (0) $\text{diam}(\pi_Y(X)) < +\infty$,
- (3) there exists ξ so that $\min\{d_Y^\pi(X, Z), d_Z^\pi(X, Y)\} \leq \xi$,
- (4) there exists ξ so that $\{Y : d_Y^\pi(X, Z) \geq \xi\}$ is a finite set for each $X, Z \in \mathbf{Y}$.

For suitably chosen constants L, K , let $\mathcal{C}(\mathbf{Y})$ be the path metric space consisting of the union of all $\mathcal{C}(Y)$ and edges connecting all points in $\pi_X(Z)$ to all points in $\pi_Z(X)$ whenever X, Z are connected by an edge in a certain complex $\mathcal{P}_K(\mathbf{Y})$ whose definition we do not need.

We are ready to state the (special case of the) result we will use.

Theorem 5.7. [BBF, Theorem 3.10, Lemma 3.1, Corollary 3.12] *If \mathbf{Y} and d^π satisfy axioms (0), (3) and (4) and each $\mathcal{C}(Y)$ is (λ, μ) -quasi-isometric to \mathbb{R} for some λ, μ not depending on Y , then $\mathcal{C}(\mathbf{Y})$ is a quasi-tree (in particular, it is hyperbolic).*

Moreover, each $\mathcal{C}(Y)$ is isometrically embedded in $\mathcal{C}(\mathbf{Y})$ and for each K there exists R so that

$$\text{diam}(N_K(\mathcal{C}(X)) \cap N_K(\mathcal{C}(Y))) \leq R$$

whenever $X \neq Y$ are elements of \mathbf{Y} .

We will use the following characterization of being hyperbolically embedded.

Theorem 5.8. [DGO, Theorem 4.39] *Let G be group and $H < G$ a subgroup. Suppose that*

- (a) G acts by isometries on a hyperbolic space S ,
- (b) there exists a quasi-convex H -orbit,
- (c) $\{H\}$ is geometrically separated, i.e. for every $K > 0$ and $s \in S$ there exists R so that if for some $g \in G$ we have $\text{diam}(Hs \cap N_K(gHs)) \geq R$ then $g \in H$,
- (d) H acts on S properly.

Then H is hyperbolically embedded in G .

Proof of Theorem 5.6. For technical reasons, in this proof we will allow projections to take values in bounded subsets of the target space. More precisely, we consider a slightly generalized definition of the set A being \mathcal{PS} -contracting with constant C where the projection map π_A is allowed to take value in subsets of A of diameter bounded by C , while properties 1) and 2) in Definition 2.2 are left unchanged. If we have a map π_A with these properties we can define a projection π'_A in the sense of Definition 2.2 just by choosing some $\pi'_A(x) \in \pi_A(x)$ for each $x \in X$. In particular our results, including Lemma 2.5 and Corollary 5.4, hold for this more general notion (possibly with different constants).

Let \mathbf{Y} be the collection of all left cosets of $E(g)$ in G and for each $Y \in \mathbf{Y}$ let $\mathcal{C}(Y)$ be a copy of $E(g)$ (more precisely, a copy of the Cayley graph of $E(g)$ with respect to a given finite set of generators), which we regard as identified with $E(g)x_0$, for some given x_0 . Each orbit $hE(g)x_0$ is a contracting set, and the constant can be chosen uniformly. Actually, we have more. In fact, there exists a collection of equivariant projections on the orbits: choose a projection π'_e on $E(g)x_0$, define $\pi_e(x) = \bigcup_{h \in E(g)} h\pi'_e(h^{-1}x)$ and define a projection on $hE(g)x_0$ by $\pi_h(x) = h\pi_e(h^{-1}x)$. In order to show that π_h is actually a projection we just

need to prove that we can uniformly bound $\text{diam}(\pi_h(x))$ for each x . This follows directly from the easily checked fact that there exist constants D_1, D_2 so that for each special path γ from x to $\pi_h(x)$ we have $\text{diam}(\gamma \cap N_{D_1}(E(g)x_0)) \leq D_2$. We can now define

$$\pi_{hE(g)}(h'E(g)) = h^{-1}\pi_h(h'E(g)x_0).$$

Axiom (0) follows from Corollary 5.4, while Lemma 2.5 implies Axiom (3). Axiom (4) is easy: if ξ is large enough then

$$|\{hE(g) : d_{hE(g)}(h_1E(g), h_2E(g)) \geq \xi\}| \leq d(h_1E(g)x_0, h_2E(g)x_0)$$

because all special paths from $h_1E(g)x_0$ to $h_2E(g)x_0$ have long disjoint subpaths each contained in some $hE(g)x_0$ for $hE(g)$ satisfying $d_{hE(g)}(h_1E(g), h_2E(g)) \geq \xi$. As all axioms are satisfied, Theorem 5.7 applies. Notice that all constructions are equivariant (including that of $\mathcal{P}_K(\mathbf{Y})$), so that there is a natural action of G on $\mathcal{C}(\mathbf{Y})$. We wish to show that this action satisfies all properties of Theorem 5.8, with $H = E(g)$ and $S = \mathcal{C}(\mathbf{Y})$. In fact, in view of Theorem 5.7, $\mathcal{C}(\mathbf{Y})$ is hyperbolic. Also, there exists an orbit of $E(g)$ which is an isometrically embedded copy of (the vertices in a Cayley graph of) $E(g)$ and $E(g)$ acts on it in the natural way, so that (b) and (d) are satisfied. Finally, (c) follows from the “moreover” part of Theorem 5.7. \square

6. RANDOM WALKS

Recall that our aim is to prove the following.

Theorem 6.1. *Let G be a group supporting the path system (resp. weak path system) (X, \mathcal{PS}) and let $H < G$ be a non-elementary finitely generated subgroup containing a contracting (resp. weakly contracting) element. Then the probability that a simple random walk $\{X_n\}$ supported on H gives rise to a non-contracting (resp. non-weakly-contracting) element decays exponentially in the length of the random walk.*

Let $\hat{g} \in H$ be (weakly) contracting and let γ_m be special paths connecting $\hat{g}^{-m}(x_0)$ to $\hat{g}^m(x_0)$ for each $m \in \mathbb{N}$.

Lemma 6.2. *There exists $K \geq 1$, $n_0 \geq 0$ and $0 < \alpha < 1$ with the following property. For each m , the probability that $d(\pi_{\gamma_m}(X_n x_0), x_0) \geq l$ is at most $K\alpha^l$ for each $n \geq n_0$.*

Proof. We will show that

$$\mathbb{P}[d(\pi_{\gamma_m}(X_n x_0), x_0) \in [l, l+K]] \geq \frac{1}{K} \mathbb{P}[d(\pi_{\gamma_m}(X_n x_0), x_0) \geq l+K], \quad (*)$$

for K large enough. This implies the lemma, as one can then show inductively that

$$\mathbb{P}(d(\pi_{\gamma_m}(X_n x_0), x_0) \geq jK) \leq \left(1 - \frac{1}{K+1}\right)^j.$$

By Lemma 5.4, we can find $K_0 \geq 0$ and an element $h \in G$ such that $\pi_{\gamma_m}(h\gamma_m) \subseteq B_{K_0}(x_0)$ for each m . We can also assume, up to increasing K_0 , that $\pi_{h\gamma_m}(\gamma_m) \subseteq B_{K_0}(x_0)$. Let $n_0 = |h|$. Set $d = d(x_0, h(x_0))$ and $d' = \max\{d(x_0, sx_0) : s \in S\}$, where S is the system of generators of H used to construct the random walk. Now, let g be a word in S . Suppose that

$$d(\pi_{\gamma_m}(gx_0), x_0) \geq l + (n_0 + 1)(kd' + k) + K_0 + L + 1, \quad (*)$$

where $L = kC + k + C$ for k as in Lemma 2.5. Now, write $g = g_0 g_1 g_2$ in such a way that

- $d(\pi_{\gamma_m}(g_0 x_0), x_0) \geq l + K_0$, and g_0 is the shortest initial subword of g with this property,
- $|g_2| = |h|$.

It is possible to find such g_i 's in view of (*) and Lemma 2.4–(2).

Now, it is easily seen using Lemma 2.5 with $A = \gamma_m$ and $B = g_0 h \gamma_m$ that, for $g' = g_0 h g_1$, we have

$$d(\pi_{\gamma_m}(g' x_0), x_0) \in [l, (l + K_0 + (kd' + k)) + K_0 + L + 1] = [l, l + K_1].$$

There is K_2 such that $\mathbb{P}[X_n = g'] \geq \mathbb{P}[X_n = g]/K_2$, and at most K_2 words g modified according to the procedure above transform into a given g' . We can now set $K = \max\{K_i^2\}$ and we are done. \square

Here is the last preliminary result, which easily follows from standard facts about random walks on non-amenable groups.

Lemma 6.3. *For each l , the probability that $\{X_n\}$ ends up in a point in $B_l(e)$ decays exponentially in n .*

Proof. We recall some notation from [Wo]. Let $p^{(n)}(e, y)$ to be the probability that the random walk $\{X_n\}$ ends up in y after n steps. The *spectral radius* of the random walk is defined as

$$\rho = \limsup p^{(n)}(e, y)^{1/n},$$

where y is any element of H (it does not depend on y). As we point out in the Appendix, H contains a free group on two generators and so it is not amenable. Hence, [Wo, Corollary 12.12] gives $\rho < 1$. Consider some $\rho < \sigma < 1$. It is quite clear from the definitions that for n large enough the probability as in the statement satisfies

$$\sum_{y \in B_l(e)} p^{(n)}(e, y) \leq |B_l(e)| \sigma^n,$$

which is what we were aiming for. \square

We are now ready to prove the theorem.

Proof of Theorem 6.1. With a slight abuse, we will sometimes regard X_n as a random variable with values in $W_n(S)$ instead of G (recall that $W_n(S)$ is the set of words of length n in elements of S and their inverses, where S is a fixed finite generating set for H). Let m be an integer to be determined later. The probability that X_n does not contain the (weakly) contracting element \hat{g}^m as a subword decays exponentially in n , let us write

$$\mathbb{P}[\hat{g}^m \not\subseteq X_n] \leq K_m \beta_m^n,$$

for some $0 < \beta < 1$ and $K_m \geq 1$. Let l be any positive real number. In view of the previous lemma and Lemma 4.1 we can estimate $\mathbb{P}[X_n \text{ is (weakly) contracting}]$ in the following way. Suppose that $\hat{g}^m \subseteq X_n$, the estimate on the probability of this happening is above. For convenience let us assume that \hat{g}^m is a final subword. We can do this as $\mathbb{P}[X_n = w_1 w_2 w_3] = \mathbb{P}[X_n = w_3 w_1 w_2]$ for any words w_1, w_2, w_3 . In view of Lemma 4.1, applied to $g = \hat{g}^m$ and h the initial subword of X_n before g , we have that X_n is (weakly) contracting if $d(\pi_{\gamma_m}(h x_0), x_0), d(\pi_{\gamma_m}(h^{-1} x_0), x_0)$ are both bounded by l (for m large enough depending on l) and $d(1, h)$ is large enough

to guarantee, together with the previous condition, that $h \notin E(g)$. (In order to apply Lemma 4.1 we first increase the projection constant C to be larger than the constant in Lemma 5.2). We can estimate the probability that this happens using, respectively, Lemma 6.2 applied to X_n and X_n^{-1} (we use that the law of the random walk is invariant under taking inverses) and Lemma 6.3. To sum up, we have constants $m(l), n_0$ such that, for $m \geq m(l)$ and $n \geq n_0$,

$$\mathbb{P}[X_n \text{ not (weakly) contracting}] \leq K_m \beta_m^n + 2K\alpha^l,$$

where we suitably increased K_m, β_m so that $K_m \beta_m^n$ bounds from above (for $n \geq n(l)$) the sum of probabilities that $\hat{g}^m \not\subseteq X_n$ and that $d(1, h)$ is smaller than required. Notice that this estimate is already good enough to show that the desired probability tends to 0, but we want a stronger result. In order to obtain it, notice that we can “iterate” the estimate, meaning that if the projections fail to be controlled, we can repeat the procedure above with h instead of X_n , i.e. we can estimate the probability that h (regarded as a random walk with fewer steps) does not contain \hat{g}^m etc.

This gives the estimate

$$\mathbb{P}[X_n \text{ not (weakly) contracting}] \leq K_m \beta_m^n + 2K\alpha^l (K_m \beta_m^{n-md} + 2K\alpha^l),$$

where d is the length of \hat{g} . We can iterate this j times if $n - jmd \geq n_0$. Iterating until we can we get

$$\mathbb{P}[X_n \text{ not (weakly) contracting}] \leq p_{\lfloor (n-n_0)/(md) \rfloor},$$

where p_j is defined inductively in the following way:

$$\begin{cases} p_1 = 2K\alpha^l, \\ p_j = K_m \beta_m^{jmd+n_0} + 2K\alpha^l p_{j-1} \text{ for } j \geq 2. \end{cases}$$

One can prove inductively that $p_j \leq (K_m \beta_m^{n_0})^j \max\{\beta_m^m d, 2K\alpha^l\}^j$, so that p_j decays exponentially, and this completes the proof. \square

APPENDIX: WEAK TITS ALTERNATIVE

In this appendix we sketch the proof of a result implied by Theorem 5.6 and results in [DGO] (see, e.g., Theorem 2.23), that is to say that subgroups containing (weakly) contracting elements are either virtually cyclic or they contain a free group on two generators. We do so, as explained in the introduction, to show that our techniques can prove Theorem 1.4 in an almost self-contained way.

Lemma 6.4. *Let g_1, g_2 be weakly contracting elements and let $\gamma_{i,m}$ be a special path connecting $g_i^{-m}x_0$ to $g_i^m x_0$. Suppose*

$$\sup\{\text{diam}(\pi_{\gamma_{1,j}}(\gamma_{2,k})), \text{diam}(\pi_{\gamma_{2,j}}(\gamma_{1,k}))\} < +\infty.$$

Then there exists n_0 such that, for each $|n|, |m| \geq n_0$, g^n and h^m form a basis of a free subgroup of G .

Proof. [Sketch] Set $Y = \bigcup_{g \in G, j \in \mathbb{N}} (g\gamma_{1,j} \cup g\gamma_{2,j}) \subseteq X$. Clearly, Y is G -invariant. Let $\pi_i : Y \rightarrow \{-N, \dots, N\}$ be such that $d(g_i^{N\pi_i(x)}x_0, \pi_{\gamma_{i,N^2}}(x))$ is minimal for each $x \in Y$. Set $A_i^\pm = \{y \in Y : \pi_i(y) > 0\}$. We just have to verify the hypotheses of the Ping Pong Lemma. Let us show, for n large enough compared to N , that $g_i^n(A_{i+1}^\pm \cup A_i^\pm) \subseteq A_i^\pm$, the analogous fact for g_i^{-n} admitting the same proof. Let

us start by showing that $g_i^n(A_i^+) \subseteq A_i^+$. Let $x \in A_i^+$ and consider a special path δ connecting x to $g_i^{N\pi_i(x)}$. We have (up to taking n large enough) that $g_i^n\delta$ cannot pass close to $\{g_i^{Nk}x_0\}_{k \leq 0}$, for otherwise δ would do the same, and this forces $\pi_i(g_i^n x) > 0$.

Let us now show $g_i^n(A_{i+1}^\pm) \subseteq A_i^+$. By Lemma 2.5 and N large enough, we have $\pi_i(A_{i+1}^\pm) = 0$, so that a special path from $x \in A_{i+1}^\pm$ to $\pi_{\gamma_i, N^2}(x)$ will stay far from each g_i^{Nk} for $k \neq 0$. Hence, an argument very similar to the one above gives that $\pi_i(g_i^n x) > 0$. \square

Theorem 6.5 (Weak Tits Alternative). *Let G be a finitely generated group equipped with the weak path system (X, \mathcal{PS}) . Let $H < G$ be a subgroup containing a weakly \mathcal{PS} -contracting element. Then either H is virtually cyclic or it contains a free group on two generators.*

The theorem above was known already in many cases. For relatively hyperbolic groups it essentially follows from the proofs in [Xi], for mapping class groups see [Iv, Mc], for $\text{Out}(F_n)$ see [IKL] (the “full” Tits Alternative is also known [BFH1, BFH2]), for right-angled Artin groups it follows from Tits’ original result [Ti] in view of the fact that they are linear [HW, DJ] (and standard facts about $CAT(0)$ spaces), and for graph manifold groups one just needs to use some Bass-Serre theory.

Proof. [Sketch] Let $g \in H$ be a weak \mathcal{PS} -contracting element and let $x_0 \in X$. If H is not virtually cyclic then we can find $h \in H$ with $h \notin E(g)$ (see Corollary 5.4). We can use Lemma 4.1 to show that hg^n is weakly contracting for some large enough n .

Now, let γ_m (resp. δ_m) be a special path connecting x_0 to $g^{mn}x_0$ (resp. x_0 to $(hg^n)^mx_0$). We wish to find a uniform bound on the diameters of $\pi_{\gamma_j}(\delta_k)$ and $\pi_{\delta_j}(\gamma_k)$, so that Lemma 6.4 gives us the wished free group. Actually, it is enough to bound just the diameter $D_{j,k}$ of, say, $\pi_{\gamma_j}(\delta_k)$, and the bound for $\pi_{\delta_j}(\gamma_k)$ will follow easily.

If $D_{j,k}$ is large enough then we would be able to find a bound on $d(hx_0, \langle g \rangle x_0)$ depending on g only, as in this case a long subpath of δ_k would travel parallel to γ_j . We can assume that such bound does not hold. \square

We point out a consequence of the Weak Tits Alternative. The following result is implicit in [Xi].

Corollary 6.6. *[Relative Tits Alternative] Let G be a torsion-free relatively hyperbolic group. Any $H < G$ either is virtually cyclic, it contains a free group on two generators or it is parabolic (i.e. contained in a conjugate of a peripheral subgroup).*

In particular, G as above satisfies the Tits alternative if and only if its peripheral subgroups do.

Proof. In view of the Weak Tits Alternative and Proposition 3.1 we only need to show that if $H < G$ does not contain a hyperbolic element then it is parabolic. This is an easy consequence of the following lemma.

Lemma 6.7. *If H_1, H_2 are peripheral subgroups, $g \in H_1$, $h = kh'k^{-1} \in kH_2k^{-1}$ for some $k \in G$, $H_1 \neq kH_2$ and $d(e, g), d(e, h')$ are large enough compared to $d(e, k)$, then gh is hyperbolic.*

Proof. We will use the path system (X, \mathcal{PS}) constructed in Proposition 3.1 and apply Lemma 4.2, with $A = H_1$. By [Si1], the projection of H on H' , whenever $H \neq H'$ are left cosets of peripheral subgroups, is uniformly bounded by, say, C , and a similar statement holds true in X . Also, we can assume $h \notin H_1$ as $\text{diam}(H_1 \cap N_{d(x_0, kx_0)}(kH_2))$ can be bounded in terms of $d(x_0, kx_0)$ by [DS, Theorem 4.1–(α_1)]. In particular $ghH_1 \neq H_1$ so that condition 2) of Lemma 4.2 is satisfied. Indeed, condition 1) is also satisfied as $d(\pi_{H_1}(ghx_0), gx_0)$ can be bounded in terms of $d(x_0, kx_0)$, as well as $d(\pi_{H_1}(h^{-1}x_0), x_0)$. \square

In order to conclude the proof notice if $H < G$ contains parabolic elements g, h not contained in the same conjugate of some peripheral subgroup, up to conjugating both of them and taking high enough powers we can reduce to the situation of the lemma (as G is torsion free), and hence we have that H contains a hyperbolic element. \square

Remark 6.8. Notice that the lemma in the proof above implies results by Osin [Os2] on the existence of hyperbolic elements.

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